

Geometry of Iteration Stable Tessellations: Connection with Poisson Hyperplanes

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Abstract

Since the seminal work [19, 20] the iteration stable (STIT) tessellations have attracted considerable interest in stochastic geometry as a natural and flexible, yet analytically tractable model for hierarchical spatial cell-splitting and crack-formation processes considered in stochastic geometry. We provide in this paper a fundamental link between typical characteristics of STIT tessellations and those of suitable mixtures of Poisson hyperplane tessellations using martingale techniques and general theory of pure jump Markov processes. As applications, new mean values and new distributional results for STIT tessellations are obtained.

Key words: Iteration/Nesting; Markov Process; Martingale Theory; Random Tessellation; Stochastic Stability; Stochastic Geometry

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1 Introduction

Infinite divisibility or stochastic stability of a random object under a certain operation is one of the most fundamental concepts in probability theory, prominent examples include the classical theory of infinite divisible and stable distributions with their applications around the central limit theorem, max-stable distributions studied in extreme value theory or union infinitely divisible random sets studied in the classical theory of random closed sets [14].

In the present paper we deal with a class of non-stationary iteration infinitely divisible random tessellations of the d -dimensional Euclidean space and, more specifically, with stationary random tessellations that are **stable** under the operation of **iteration** – called STIT tessellations for short. Recall that a tessellation (or mosaic) of \mathbb{R}^d is a locally finite family of compact convex random polytopes, which have pairwise no common interior points, cover the whole space and that they are a central object studied in stochastic geometry and related fields, see [1, 2, 4, 6, 7, 8, 9] to name just a few.

In the stationary case, the motivation for random STIT tessellations goes back to R. Ambartzumian in the 80thies and the principle of iteration of tessellations can roughly be

explained as follows: Take a random primary or frame tessellation and associate with each of its cells independently a copy of the primary tessellation, called component tessellation, which is also independent of the primary tessellation as well. Perform now in each cell a local superposition of the primary tessellation and the associated tessellation, whereupon scale the resulting random tessellation by a factor 2 in order to ensure that the mean surface measure of cell boundaries stays constant. The described operation can be repeatedly applied and we obtain in this way a sequence of random tessellations. It can be shown that this sequence converges to a random limit tessellation and that this tessellation is stable under iterations – a STIT tessellation – in the translation-invariant set-up. In the general case, the resulting tessellation is iteration infinitely divisible in any finite volume.

Starting with [19], STIT tessellations and their theoretical framework were formally introduced in [20] by Nagel and Weiss. The same research group discovered a first basic technique for studying mean values and even some distributions related to the geometry of STIT tessellations by writing certain balance equations based on the stochastic stability of the tessellation, see [21, 22, 32] for the mean values as well as [15, 16] for distributional results. In [17, 18] a new aspect was introduced into the theory, namely a tessellation-valued random Markov process on the positive real half-axis with the property that at each time the law of the tessellation is stable under iteration. In finite compact convex volumes $W \subset \mathbb{R}^d$ with $\text{Vol}_d(W) > 0$, this process can be explained as follows: Let us fix a terminal time $t > 0$ and a (in some sense non-degenerate) translation-invariant measure Λ on the space of hyperplanes in \mathbb{R}^d . In a first step, we assign to W a random lifetime. Upon expiry of its lifetime, the primordial cell W dies and splits into two polyhedral sub-cells separated by a hyperplane hitting W , which is chosen according to the normalized distribution Λ . The resulting new cells are again assigned independent random lifetimes and the entire construction continues recursively until the deterministic time threshold t is reached (see Figure 1 for an illustration). The resulting random tessellation is denoted by $Y(t\Lambda, W)$. In order to ensure the Markov property of the above construction in the continuous-time parameter t and in order to have distributional equality of the outcome with the above mentioned limit tessellations, we will have to take care of the special choice of the lifetime distributions, see the details in Section 3 below.

This dynamic point of view can be exploited to establish new results on STIT tessellations and in fact our theory will be based on it. In addition, the finite volume Markovian construction provides a link to a class of more general and non-stationary iteration infinitely divisible random MNW-tessellations, where the translation-invariant hyperplane measure Λ from above is replaced by an arbitrary non-atomic and locally finite measure on the space of hyperplanes. We would like to emphasize that the dynamical representation is a special feature of STIT tessellations and their infinitely divisible counterparts and, as recently pointed out by the first author [25], that such and similar spatio-temporal random processes have remarkable potential for applications in stochastic geometry.

The purpose of the present paper is to explore the dynamic representation of STIT tessellations further and to introduce a new technique, which has the advantage to deal with properties of the model that were out of reach so far. The crucial observation is that the dynamic representation of the tessellations under consideration fulfils a Markov

property in the continuous-time parameter, which allows to construct certain martingales related to the tessellations (see Section 4), eventually leading to fundamental comparison results of STIT tessellations with certain mixtures of Poisson hyperplane tessellations in Section 5. As an application of these results we calculate new mean values and also some new distributions related to geometric objects determined by the tessellation. To keep the paper self-contained, we recall in Section 2 the construction of STIT tessellations as limit of repeated iterations of tessellations and formally introduce in Section 3 their Markovian dynamic representation. Moreover, we also introduce there the class of iteration infinitely divisible random MNW-tessellations and summarize some of their most important properties needed in this paper.

The current work is based on an extended version available online [26]. It also forms the basis of our subsequent papers [27, 28, 29, 30].

2 STIT tessellations as limits

A tessellation of \mathbb{R}^d is a locally finite partition of the space into compact convex polytopes, the cells of the tessellation. One can regard a tessellation either as a collection of its cells or as the closed set formed by the union of their boundaries. We will mostly follow the second mentioned point of view and denote by $\text{Cells}(Y)$ the set of cells of the tessellation Y (note that by the Jordan–Schönflies theorem the correspondence between Y and $\text{Cells}(Y)$ is one-to-one). Thus, a random tessellation can be regarded as a special random closed set in the classical sense of stochastic geometry, see [14, 23, 24]. In particular, this imposes the usual Fell topology and the corresponding Borel measurable structure on the family of tessellations, see *ibidem*. A random tessellation Y (regarded as a random closed set in \mathbb{R}^d) is called *stationary* if its distribution does not change upon actions of translations. Analogously a random tessellation is called *isotropic* if its distribution is invariant under the action of $SO(d)$.

Whenever we two random tessellations Y_1 and Y_2 of \mathbb{R}^d are given we can define their *iteration/nesting*. To this end, we associate to each cell $c \in \text{Cells}(Y_1)$ an independent copy $Y_2(c)$ of Y_2 and we assume furthermore the family $\{Y_2(c) : c \in \text{Cells}(Y_1)\}$ to be independent of Y_1 . Then we define the iteration of Y_1 with Y_2 by

$$Y_1 \boxplus Y_2 := Y_1 \boxplus \{Y_2(c) : c \in \text{Cells}(Y_1)\} := Y_1 \cup \bigcup_{c \in \text{Cells}(Y_1)} (Y_2(c) \cap c),$$

i.e. we take the local superposition of Y_1 and the family $\{Y_2(c) : c \in \text{Cells}(Y_1)\}$ inside the cells of Y_1 . It was shown in [18] that $Y_1 \boxplus Y_2$ is a stationary random tessellation as soon Y_1 and Y_2 have this property. Note also that the so-defined nesting operation is associative in distribution. A stationary random tessellation Y is called *stable under iteration*, or *STIT* for short, iff

$$m \underbrace{(Y \boxplus \dots \boxplus Y)}_m \stackrel{D}{=} Y, \quad m = 2, 3, \dots, \quad (1)$$

where $\stackrel{D}{=}$ stands for equality in distribution (note that rescaling by factor m ensures that the mean surface area of cell boundaries per unit volume remains constant). In fact, using the uniqueness results, see Theorem 3 and Corollary 2 in [20], it is easy to see that it is enough to take one fixed $m > 1$ in (1).

To proceed, let us be given a constant $0 < t < \infty$ and a probability measure \mathcal{R} on the unit sphere \mathcal{S}_{d-1} , usually identified with the induced distribution of orthogonal hyperplanes on the space of $(d-1)$ -dimensional linear hyperplanes in \mathbb{R}^d , also denoted by \mathcal{R} in the sequel for notational simplicity. Define the measure Λ on the space \mathcal{H} of affine hyperplanes in \mathbb{R}^d as the product measure

$$\Lambda := \ell_+ \otimes \mathcal{R} \quad (2)$$

of ℓ_+ standing for the Lebesgue measure on the positive real half-axis $(0, \infty)$, and of \mathcal{R} , where a pair $(r, u) \in (0, \infty) \times \mathcal{S}_{d-1}$ is identified with the hyperplane $\{x \in \mathbb{R}^d, \langle x, u \rangle = r\}$. Throughout this paper we always require that the support of \mathcal{R} spans the whole space, i.e. that $\text{span}(\text{supp}(\mathcal{R})) = \mathbb{R}^d$. Assume now that we are given a stationary random tessellation Y with surface intensity t (i.e. the mean surface area of cell boundaries per unit volume equals t) and directional distribution \mathcal{R} (i.e. the distribution of the normal direction of the facet containing the typical point in the standard Palm sense is given by \mathcal{R}) and define the sequence $(\mathcal{I}_n(Y))_{n \geq 1}$ by

$$\mathcal{I}_1(Y) := 2(Y \boxplus Y), \quad \mathcal{I}_n(Y) := \frac{n}{n-1} \mathcal{I}_{n-1}(Y) \boxplus nY = n \underbrace{(Y \boxplus \dots \boxplus Y)}_n, \quad n \geq 2.$$

It was shown in [20, Thm. 3] that $\mathcal{I}_n(Y)$ converges in law, as $n \rightarrow \infty$, to a stationary random limit tessellation $Y(t\Lambda)$ *uniquely determined* by $t\Lambda$. This tessellation is easily shown to be stable under iterations, whence a STIT tessellation with parameters t and Λ and we refer to Figure 1 for an illustration.

3 The MNW-construction

As already emphasized in the Introduction, it is a crucial feature of the random tessellations $Y(t\Lambda)$ introduced in the previous section that they admit a natural and intuitive spatio-temporal construction. For a restriction $Y(t\Lambda, W)$ of $Y(t\Lambda)$ to a compact convex window $W \subset \mathbb{R}^d$ with interior points, this construction can be described as follows (the reader is referred to [20] for full details): Assign to the window W an exponentially distributed random lifetime with parameter $\Lambda([W])$ where $[W] := \{H \in \mathcal{H}, H \cap W \neq \emptyset\}$ stands for the family of all hyperplanes hitting W . Upon expiry of its lifetime, the cell W dies and splits into two sub-cells W^+ and W^- separated by a hyperplane in $[W]$ chosen according to the law $\Lambda(\cdot)/\Lambda([W])$. The resulting new cells W^+ and W^- are again assigned independent exponential lifetimes with respective parameters $\Lambda([W^+])$ and $\Lambda([W^-])$ (whence smaller cells live stochastically longer) and the entire construction continues recursively until the deterministic time threshold t is reached. The cell-separating $(d-1)$ -dimensional facets (the

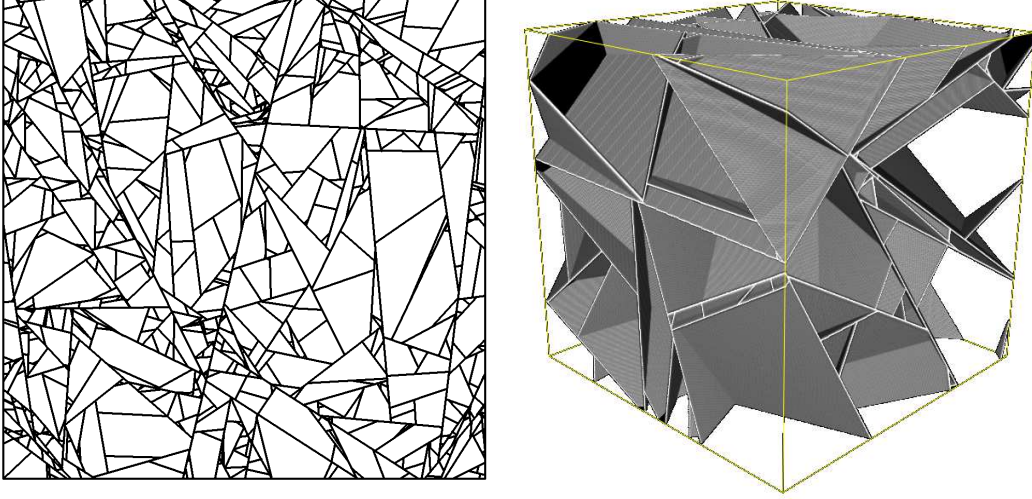


Figure 1: Realizations of a planar and a spatial stationary and isotropic STIT tessellation (kindly provided by Joachim Ohser and Claudia Redenbach)

word *facet* stands for a $(d-1)$ -dimensional face here and throughout) arising in subsequent splits are usually referred to as $(d-1)$ -dimensional *maximal polytopes* (or *I-segments* for $d=2$ as assuming shapes similar to the letter *I*). The described process of recursive cell divisions is called the Mecke-Nagel-Weiss- or *MNW-construction* in the sequel and the resulting random tessellation created inside W is denoted by $Y(t\Lambda, W)$ as mentioned above, whereas the collection of all $(d-1)$ -dimensional maximal polytopes or *I-segments* is denoted by $\text{MaxPolytopes}_{d-1}(Y(t\Lambda, W))$. Moreover, we write $\text{MaxPolytopes}_k(Y(t\Lambda, W))$ for the collection of k -dimensional maximal polytopes of $Y(t\Lambda, W)$, where by a k -dimensional maximal polytope we mean the maximal union of connected and k -coplanar k -dimensional faces of cells. In fact, the k -dimensional maximal polytopes of $Y(t\Lambda, W)$ are nothing than the k -faces of $(d-1)$ -dimensional maximal polytopes.

It was shown in [20] that the law of $Y(t\Lambda, W)$ is consistent in that $Y(t\Lambda, W) \cap V \stackrel{D}{=} Y(t\Lambda, V)$ for convex $V \subset W$ and thus $Y(t\Lambda, W)$ can be extended to random tessellation $Y(t\Lambda)$ on the whole space, which is then proved (see [20]) to coincide with the limit tessellation $Y(t\Lambda)$ considered in the previous section, as the notation suggests. Again, the sets of all $(d-1)$ -dimensional and k -dimensional maximal polytopes of $Y(t\Lambda)$ are denoted by $\text{MaxPolytopes}_{d-1}(Y(t\Lambda))$ and $\text{MaxPolytopes}_k(Y(t\Lambda))$ ($0 \leq k \leq d-2$), respectively. The stationary random tessellation $Y(t\Lambda)$ is additionally isotropic if and only if \mathcal{R} is the uniform distribution on \mathcal{S}_{d-1} in the factorization (2).

A simple yet crucial observation is that even though only translation-invariant measures Λ of the form (2) show up in the limiting STIT tessellations, the MNW-construction can be carried out with arbitrary non-atomic and locally finite driving measure Λ (i.e. $\Lambda([W]) < \infty$

for $W \subset \mathbb{R}^d$ bounded) on \mathcal{H} also leading to a consistent family $Y(t\Lambda, W)$ and eventually, by extension, yielding $Y(t\Lambda)$. Many of our theorems will be stated in this general context. It should be emphasized though that such tessellations are no longer iteration stable (STIT). However, they have the general property of being *iteration infinitely divisible*, as they can be readily checked to arise as m -fold iterations of $Y(t/m)$ for each $m \geq 2$ in all finite volumes. Formally, this means that

$$Y((t/m)\Lambda, W)^{\boxplus m} \stackrel{D}{=} Y(t\Lambda, W)$$

for all compact convex sets $W \subseteq \mathbb{R}^d$, which follows directly by the MNW-construction as yielding

$$Y(s\Lambda, W) \boxplus Y(u\Lambda, W) \stackrel{D}{=} Y((s+u)\Lambda, W).$$

It is more than natural to expect that also $Y(t\Lambda) = Y((t/m)\Lambda)^{\boxplus m}$ for all m in the whole of \mathbb{R}^d , which should be provable by adopting the theory developed in [18], thus even better justifying the term *iteration infinitely divisible* in our context, yet this falls beyond the scope of the present work and is not needed in our arguments.

It is worth pointing out that it is currently an open problem whether *any* iteration infinitely divisible random tessellation in \mathbb{R}^d can be constructed with the MNW-algorithm. This means that in this paper infinite divisibility with respect to iteration only refers to such tessellations which can be obtained by the MNW-construction. For this reason, the tessellations we are dealing with in the general set-up are called iteration infinitely divisible MNW-tessellations.

4 Associated martingales

The finite volume continuous-time incremental MNW-construction of iteration infinitely divisible random tessellations or more specially stationary STIT tessellations, as discussed in Section 3 above, clearly enjoys the Markov property in the time parameter, whence natural martingales arise, which will be of crucial importance for our further considerations. To discuss these processes we need some additional terminology and notational conventions. Let us fix a compact convex window W and a general diffuse (non-atomic) and locally finite measure Λ on \mathcal{H} , the space of hyperplanes in \mathbb{R}^d . Next, for a tessellation Y of \mathbb{R}^d and any $H \in [W]$, we denote by $\text{Cells}(Y \cap H)$ the set of cells the hyperplane H is tessellated with as a result of the intersection with Y . In terms of the MNW-construction, the members of $\text{Cells}(Y \cap H)$, which are $(d-1)$ -dimensional polytopes, are the potentially new facets to be added to the tessellation when the construction is continued. Whenever a new facet $f \in \text{Cells}(Y \cap H)$ to be added to Y should a facet birth (cell split) occur on H in Y , we denote by $\text{Cell}(f, H|Y)$ the cell splitting by the birth of f and we write $\text{Cell}^+(f, H|Y)$ and $\text{Cell}^-(f, H|Y)$ for the two sub-cells into which $\text{Cell}(f, H|Y)$ divides, lying on the positive and negative sides of H , respectively.

With this notation, it is readily seen that for a fixed measure Λ , $(Y(t\Lambda, W))_{t \geq 0}$ is a pure

jump Markov process with values in the space of tessellations in W and with the property that

$$d\mathbb{P}(Y((t+dt)\Lambda) = Y \cup \{f\} | Y(t\Lambda) = Y) = \mathbf{1}[f \in \text{Cells}(Y \cap H)]\Lambda(dH)dt, \quad (3)$$

with $H \in [W]$, and

$$\begin{aligned} \mathbb{P}(Y((t+dt)\Lambda) = Y | Y(t\Lambda) = Y) &= 1 - \left(\int_{[W]} |\text{Cells}(Y \cap H)|\Lambda(dH) \right) dt \\ &= 1 - \left(\sum_{c \in \text{Cells}(Y)} \Lambda([c]) \right) dt, \end{aligned} \quad (4)$$

where $|\text{Cells}(Y \cap H)|$ is the cardinality of the set $\text{Cells}(Y \cap H)$. Indeed, this is because, conditionally on $Y(t\Lambda) = Y$, during the period $(t, t+dt]$ of the MNW-construction we have, as discussed in Section 3:

- For each cell $c \in \text{Cells}(Y)$ the probability that it undergoes a split is $\Lambda([c])dt$, moreover the probability that two or more cells split is $o(dt)$, whence the probability that no split occurs is $1 - \left(\sum_{c \in \text{Cells}(Y)} \Lambda([c]) \right) dt$ as in (4). Noting that $\sum_{c \in \text{Cells}(Y)} \Lambda([c]) = \int_{[W]} |\text{Cells}(Y \cap H)|\Lambda(dH)$ yields the remaining equality in (4).
- Should a cell $c \in \text{Cells}(Y)$ split, the splitting hyperplane H is chosen according to the law $\mathbf{1}[H \in [c]](\cdot)/\Lambda([c])$, whence the probability of observing a split of c induced by H during the time period $(t, t+dt]$ is just $\mathbf{1}[H \in [c]](dH)dt$. Thus, we see that on the event $Y(t\Lambda) = Y$, having $Y((t+dt)\Lambda) = Y \cup \{f\}$ with $f \subset H$, $H \in [c]$ and $c \in \text{Cells}(Y)$ is equivalent to having c split by H during $(t, t+dt]$. As noted above, the latter happens with probability $\Lambda(dH)dt$ whence (3) follows.

It should be emphasized at this point that the *differential* notation in (3) and (4) employing the symbols $d\mathbb{P}$ and dt is widely accepted in the theory of pure jump continuous-time Markov processes and makes perfect formal sense as a commonly recognized abbreviation for the usual description in terms of waiting times, with $d\mathbb{P}(Y((t+dt)\Lambda) = Y' | Y(t\Lambda) = Y) = a(Y, Y')dt$ understood as '*while in state Y , wait an exponential time with parameter $a(Y, Y')$ and then jump to Y' unless some other jump has occurred prior to that*'. We refer the reader to Chapter 15 in [5] and especially to Section 15.6 there, where this construction is formalized with full mathematical rigour in terms of waiting times as noted above. As readily verified, specializing the generic waiting-time construction to the case of (3) and (4) yields precisely the standard MNW construction from Section 3.

Using (3) and (4) we conclude by general theory of Markov processes and their infinitesimal generators, see Chapter 1 in [13] or Chapter 15 and especially Sections 15.4 (Def. 15.21) and 15.6 in [5] specialized for the pure jump case, the generator for $(Y(t\Lambda, W))_{t \geq 0}$ is $\mathbb{L} := \mathbb{L}_{\Lambda; W}$ with

$$\mathbb{L}F(Y) = \int_{[W]} \sum_{f \in \text{Cells}(Y \cap H)} [F(Y \cup \{f\}) - F(Y)]\Lambda(dH) \quad (5)$$

for all F bounded and measurable on space of tessellations of W . Consequently, again by standard theory as given in Lemma 5.1 Appendix 1 Sec. 5 in [11], see also Section 1.5 in [13], or alternatively by a direct check straightforward in the present set-up, we readily conclude

Proposition 1 *For F bounded and measurable, the stochastic process*

$$F(Y(t\Lambda, W)) - \int_0^t \mathbb{L}F(Y(s\Lambda, W))ds \quad (6)$$

is a martingale with respect to the filtration \mathfrak{F}_t generated by $(Y(s\Lambda, W))_{0 \leq s \leq t}$.

To proceed towards the crucial Proposition 2, consider F of the form

$$\Sigma_\phi(Y) := \sum_{f \in \text{MaxPolytopes}_{d-1}(Y)} \phi(f)$$

where, recall, $\text{MaxPolytopes}_{d-1}(Y)$ are the $(d-1)$ -dimensional maximal polytopes of Y (the I-segments in the two-dimensional case), whereas $\phi(\cdot)$ is a generic bounded and measurable functional on $(d-1)$ -dimensional facets in W , that is to say a bounded and measurable function on the space of closed $(d-1)$ -dimensional polytopes in W , possibly chopped off by the boundary of W , with the standard measurable structure inherited from space of closed sets in W . Whereas the so-defined F is not bounded and thus (6) cannot be applied directly, we can apply it for its truncation $F_N := (F \wedge N) \vee -N$, $N \in \mathbb{N}$, which is bounded, and let $N \rightarrow \infty$ to conclude that $F(Y(t\Lambda, W)) - \int_0^t \mathbb{L}F(Y(s\Lambda, W))ds$ is a local \mathfrak{F}_t -martingale, see Definition 5.15 in [12] and take $T_N = \inf\{t \geq 0 : |F(Y(t\Lambda, W))| \geq N\}$ there. Now, apply the proof of Lemma 1 in [20], where the number of cells in $Y(t\Lambda, W)$, and hence for all $Y(s\Lambda, W)$, $s \leq t$, is bounded by a Furry-Yule-type linear birth process whose cardinality at any given finite time admits moments of all orders, to conclude that $(F(Y(t\Lambda, W)) - \int_0^t \mathbb{L}F(Y(s\Lambda, W))ds)_{t \leq a}$ is of class DL for all $a > 0$ in the sense of Definition 4.8 in [12]. Using now Problem 5.19 (i) in [12] we finally conclude that $F(Y(t\Lambda, W)) - \int_0^t \mathbb{L}F(Y(s\Lambda, W))ds$ is a martingale. Thus, applying (5) and (6) for $F \equiv \Sigma_\phi$ we have

Proposition 2 *The stochastic process*

$$\Sigma_\phi(Y(t\Lambda, W)) - \int_0^t \int_{[W]} \sum_{f \in \text{Cells}(Y(s\Lambda, W) \cap H)} \phi(f) \Lambda(dH) ds$$

is a martingale with respect to \mathfrak{F}_t .

5 Relationships for intensity measures

In this section we establish a two fundamental first-order properties of $Y(t\Lambda, W)$ for general locally finite non-atomic measures Λ , essentially obtained by comparison with suitable

mixtures of Poisson hyperplane tessellations. The key to our results is Proposition 2 from Section 4. To exploit it, consider the random measures

$$\mathcal{M}^{Y(t\Lambda, W)} := \sum_{c \in \text{Cells}(Y(t\Lambda, W))} \delta_c, \quad \mathbb{M}^{Y(t\Lambda, W)} := \mathbb{E} \mathcal{M}^{Y(t\Lambda, W)} \quad (7)$$

with δ_c standing for the unit mass Dirac measure at c . In full analogy, define $\mathcal{M}^{\text{PHT}(t\Lambda, W)}$ and $\mathbb{M}^{\text{PHT}(t\Lambda, W)}$, where $\text{PHT}(t\Lambda, W)$ is the Poisson hyperplane tessellation with intensity measure $t\Lambda$, restricted to W (see [23] for background material on Poisson hyperplane tessellations). Further, put

$$\mathcal{F}_k^{Y(t\Lambda, W)} := \sum_{f \in \text{MaxPolytopes}_k(Y(t\Lambda, W))} \delta_f, \quad \mathbb{F}_k^{Y(t\Lambda, W)} := \mathbb{E} \mathcal{F}_k^{Y(t\Lambda, W)}, \quad k = 1, \dots, d-1,$$

where, recall, $\text{MaxPolytopes}_k(Y)$ is the collection of k -dimensional maximal polytopes of Y . Likewise, define

$$\mathcal{F}_k^{\text{PHT}(t\Lambda, W)} := \sum_{f \in \text{Faces}_k(\text{PHT}(t\Lambda, W))} \delta_f, \quad \mathbb{F}_k^{\text{PHT}(t\Lambda, W)} := \mathbb{E} \mathcal{F}_k^{\text{PHT}(t\Lambda, W)}, \quad k = 1, \dots, d-1,$$

where $\text{Faces}_k(\text{PHT}(t\Lambda, W))$ is the collection of all k -face of the Poisson hyperplane tessellation $\text{PHT}(t\Lambda, W)$. Our first claim is

Theorem 1 *We have*

$$\mathbb{M}^{Y(t\Lambda, W)} = \mathbb{M}^{\text{PHT}(t\Lambda, W)}.$$

It is worth pointing out that in the translation-invariant set-up, we obtain as a corollary that the distribution \mathbb{Q} of the typical cell of the STIT tessellation $Y(t\Lambda)$ coincides with the typical cell distribution $\mathbb{Q}^{\text{PHT}(t\Lambda)}$ of a stationary Poisson hyperplane tessellation $\text{PHT}(t\Lambda)$ with intensity measure $t\Lambda$, previously shown in [19] by completely different arguments. Recall, that the typical cell of a tessellation is a randomly selected cell out of a large observation window, where each cell has the same chance of being selected. Indeed, we have by the very definition of the distributions \mathbb{Q} and $\mathbb{Q}^{\text{PHT}(t\Lambda)}$ (cf. [23, Eq. (4.8, 4.9)]) and by Campbell's theorem [23, Thm. 3.1.2] for any measurable subset A of the space of d -dimensional polytopes

$$\begin{aligned} \lambda_{\text{Cells}(Y(t\Lambda))} \mathbb{Q}(A) &= \lim_{r \rightarrow \infty} \frac{1}{r^d \text{Vol}_d(W)} \mathbb{E} \int \mathbf{1}[c - m(c) \in A] \mathcal{M}^{Y(t\Lambda, rW)}(dc) \\ &= \lim_{r \rightarrow \infty} \frac{1}{r^d \text{Vol}_d(W)} \int \mathbf{1}[c - m(c) \in A] \mathbb{M}^{Y(t\Lambda, rW)}(dc) \\ &= \lim_{r \rightarrow \infty} \frac{1}{r^d \text{Vol}_d(W)} \int \mathbf{1}[c - m(c) \in A] \mathbb{M}^{\text{PHT}(t\Lambda, rW)}(dc) \\ &= \lambda_{\text{Cells}(\text{PHT}(t\Lambda))} \mathbb{Q}^{\text{PHT}(t\Lambda)}(A), \end{aligned}$$

where $\lambda_{\text{Cells}(Y(t\Lambda))}$ and $\lambda_{\text{Cells}(\text{PHT}(t\Lambda))}$ denote, respectively, the mean number of cells of $Y(t\Lambda)$ and $\text{PHT}(t\Lambda)$ per unit volume, $W \subset \mathbb{R}^d$ is measurable with positive volume, and where

$m(c)$ stands for a translation-covariant selector for the d -dimensional polytope c (for example its Steiner point or its center of gravity). Since trivially $\lambda_{\text{Cells}(Y(t\Lambda))} = \lambda_{\text{Cells}(\text{PHT}(t\Lambda))}$ by Theorem 1, we get $\mathbb{Q}^{Y(t\Lambda)} = \mathbb{Q}^{\text{PHT}(t\Lambda)}$.

Proof of Theorem 1 Using (5) and (6) with

$$F(Y) := \sum_{c \in \text{Cells}(Y)} \phi(c)$$

for general bounded measurable cell functional ϕ , with localization argument as the one preceding Proposition 2 we conclude that

$$\begin{aligned} & \int \phi(c') \mathcal{M}^{Y(t\Lambda, W)}(dc') - \int_0^t \int_{[W]} \sum_{f \in \text{Cells}(Y(s\Lambda, W) \cap H)} [\phi(\text{Cell}^+(f, H|Y(t\Lambda, W))) + \\ & \quad \phi(\text{Cell}^-(f, H|Y(t\Lambda, W))) - \phi(\text{Cell}(f, H|Y(t\Lambda, W)))] \Lambda(dH) ds \end{aligned} \quad (8)$$

is a \mathfrak{S}_t -martingale. For a polyhedral cell $c \subseteq W$, possibly chopped off by the boundary of W , and for $H \in [c]$ we write $c^+(H)$ and $c^-(H)$ to denote the cells into which c gets divided by H , lying respectively on the positive and negative side of H . With this notation, (8) says that

$$\int \phi(c') \mathcal{M}^{Y(t\Lambda, W)}(dc') - \int_0^t \int \int_{[c]} [\phi(c^+(H)) + \phi(c^-(H)) - \phi(c)] \Lambda(dH) \mathcal{M}^{Y(s\Lambda, W)}(dc) ds$$

is a \mathfrak{S}_t -martingale. Taking expectations leads to

$$\int \phi(c') \mathbb{M}^{Y(t\Lambda, W)}(dc') = \int_0^t \int \int_{[c]} [\phi(c^+(H)) + \phi(c^-(H)) - \phi(c)] \Lambda(dH) \mathbb{M}^{Y(s\Lambda, W)}(dc) ds \quad (9)$$

for all bounded measurable ϕ . To proceed, we regard $\mathbb{M}^{Y(s\Lambda, W)}$ as an element of the space of bounded variation Borel measures on the family of polyhedral sub-cells of W endowed with the standard measurable structure inherited from the space of closed sets in W . Consider the linear operator T_Λ on this measure space, given by

$$T_\Lambda(\mu) = \int \int_{[c]} [\delta_{c^+(H)} + \delta_{c^-(H)} - \delta_c] \Lambda(dH) \mu(dc). \quad (10)$$

By the definition (10), $\|T_\Lambda(\mu)\|_{\text{TV}} \leq (\int_{[W]} d\Lambda) \|\mu\|_{\text{TV}} = \Lambda([W]) \|\mu\|_{\text{TV}}$ where $\|\cdot\|_{\text{TV}}$ is the standard total variation norm of a measure, see [3, Def. 3.1.4]. This inequality turns into equality when $\mu = \delta_W$. Consequently, T_Λ is a bounded operator of operator norm $\Lambda([W]) < +\infty$ by the assumed locally finiteness of Λ . Using the operator T_Λ , the relation (9) can be rewritten in form of the operator differential equation

$$\frac{\partial}{\partial t} \mathbb{M}^{Y(t\Lambda, W)} = T_\Lambda \mathbb{M}^{Y(t\Lambda, W)}, \quad \mathbb{M}^{Y(0, W)} = \delta_W, \quad (11)$$

which, in view of the above properties of T_Λ , admits by standard theory of linear operators (cf. [10, IX.§2, Sec. 2]) the unique solution

$$\mathbb{M}^{Y(t\Lambda, W)} = \exp(tT_\Lambda)\delta_W, \quad t \geq 0. \quad (12)$$

It is easily seen that exactly the same equations (9), (11) and thus also (12) hold for $\mathbb{M}^{\text{PHT}(t\Lambda, W)}$. In particular, $\mathbb{M}^{Y(t\Lambda, W)} = \mathbb{M}^{\text{PHT}(t\Lambda, W)}$ as required. \square

Having characterized $\mathbb{M}^{Y(t\Lambda, W)}$ we now turn to $\mathbb{F}_k^{Y(t\Lambda, W)}$.

Theorem 2 *For all $k = 1, \dots, d-1$ we have*

$$\mathbb{F}_k^{Y(t\Lambda, W)} = (d-k)2^{d-k-1} \int_0^t \frac{1}{s} \mathbb{F}_k^{\text{PHT}(s\Lambda, W)} ds.$$

Proof of Theorem 2 Fix $k \in \{1, \dots, d-1\}$. Let ψ be a general bounded measurable function of a k -dimensional maximal polytope, as usual regarded as a closed subset of W , and for a $(d-1)$ -dimensional maximal polytope h put

$$\phi(h) := \sum_{f \in \text{Faces}_k(h)} \psi(f), \quad (13)$$

noting that the k -dimensional maximal polytopes of the tessellation $Y(t\Lambda, W)$ are precisely the k -faces of its $(d-1)$ -dimensional maximal polytopes. Using Proposition 2, taking expectations and recalling (7) we see that

$$\mathbb{E}\Sigma_\phi(Y(t\Lambda, W)) = \int \phi d\mathbb{F}_{d-1}^{Y(t\Lambda, W)} = \int_0^t \int \int_{[c]} \phi(c \cap H) \Lambda(dH) \mathbb{M}^{Y(s\Lambda, W)}(dc) ds.$$

Applying Theorem 1 we get

$$\int \phi d\mathbb{F}_{d-1}^{Y(t\Lambda, W)} = \int_0^t \int \int_{[c]} \phi(c \cap H) \Lambda(dH) \mathbb{M}^{\text{PHT}(s\Lambda, W)}(dc) ds. \quad (14)$$

However, by Slivnyak's theory, see e.g. [24, Thm 1.15], we obtain

$$\int \int_{[c]} \phi(c \cap H) \Lambda(dH) \mathbb{M}^{\text{PHT}(\Lambda, W)}(dc) = \int \phi d\mathbb{F}_{d-1}^{\text{PHT}(\Lambda, W)}$$

and thus, upon taking $s\Lambda$ in place of Λ , more generally,

$$\int \int_{[c]} \phi(c \cap H) \Lambda(dH) \mathbb{M}^{\text{PHT}(s\Lambda, W)}(dc) = \frac{1}{s} \int \phi d\mathbb{F}_{d-1}^{\text{PHT}(s\Lambda, W)},$$

whence, with (14),

$$\int \phi d\mathbb{F}_{d-1}^{Y(t\Lambda, W)} = \int_0^t \frac{1}{s} \int \phi d\mathbb{F}_{d-1}^{\text{PHT}(s\Lambda, W)} ds.$$

We note now that, by (13),

$$\int \phi d\mathbb{F}_{d-1}^{Y(t\Lambda, W)} = \int \psi d\mathbb{F}_k^{Y(t\Lambda, W)},$$

because each k -dimensional maximal polytope is a k -face of precisely one $(d-1)$ -dimensional maximal polytope in $Y(t\Lambda, W)$. On the other hand,

$$\int \phi d\mathbb{F}_{d-1}^{\text{PHT}(s\Lambda, W)} = (d-k)2^{d-k-1} \int \psi d\mathbb{F}_k^{\text{PHT}(s\Lambda, W)},$$

because each k -face of $\text{PHT}(s\Lambda, W)$ is a k -face of $(d-k)2^{d-k-1}$ facets of $\text{PHT}(s\Lambda, W)$, see Theorems 10.1.2 and 10.3.1 in [23]. Hence, we conclude that

$$\int \psi d\mathbb{F}_k^{Y(t\Lambda, W)} = (d-k)2^{d-k-1} \int_0^t \frac{1}{s} \int \psi d\mathbb{F}_k^{\text{PHT}(s\Lambda, W)} ds$$

for all ψ bounded and measurable, which completes the proof of the Theorem. \square

Some of our arguments in the sequel will require a straightforward formal extension of Theorem 2. Namely, we formally *mark* all $(d-1)$ -dimensional maximal polytopes of the tessellation $Y(t\Lambda, W)$ by their *birth times*. This gives rise to the birth-time augmented tessellation $\hat{Y}(t, W)$ with birth-time-marked $(d-1)$ -dimensional maximal polytopes and makes the MNW-construction of $\hat{Y}(t, W)$ into a Markov process whose generator $\hat{\mathbb{L}}$ is a clear modification of \mathbb{L} as given in (5):

$$\hat{\mathbb{L}}\hat{F}(\hat{Y}) = \int_{[W]} \sum_{f \in \text{Cells}(Y \cap H)} [\hat{F}(\hat{Y} \cup [\{f\}, s]) - \hat{F}(\hat{Y})] \Lambda(dH)$$

for \hat{F} bounded and measurable on the space of birth time-marked tessellations of W . Consequently, writing $\hat{\mathbb{F}}_k^{Y(t\Lambda, W)}$, $k = 1, \dots, d-1$, for the birth-time-marked version of $\mathbb{F}_k^{Y(t\Lambda, W)}$, where each k -dimensional maximal polytope is marked with its birth time, by a straightforward modification of the proof of Theorem 2 we are led to

Corollary 1 *For all $k = 1, \dots, d-1$ we have*

$$\hat{\mathbb{F}}_k^{Y(t\Lambda, W)} = (d-k)2^{d-k-1} \int_0^t \frac{1}{s} \left[\mathbb{F}_k^{\text{PHT}(s\Lambda, W)} \otimes \delta_s \right] ds.$$

6 Typical maximal polytope distributions

We are now going to apply the results obtained in the last section to the stationary set-up, i.e. with Λ translation-invariant, to study the distribution of typical k -dimensional maximal polytopes of the STIT tessellation $Y(t\Lambda)$ in \mathbb{R}^d . Recall, that, in intuitive terms, the typical k -dimensional maximal polytope of $Y(t\Lambda)$ is what we get when we equiprobably choose a k -dimensional maximal polytope of the tessellation out of a large observation window. This intuition can be made precise either by using Palm calculus or an ergodic approach, see [23, 24] for details.

Theorem 3 *The distribution \mathbb{Q}_k of the typical k -dimensional maximal polytope of $Y(t\Lambda)$ is given by*

$$\mathbb{Q}_k = \int_0^t \frac{ds^{d-1}}{t^d} \mathbb{Q}_k^{\text{PHT}(s\Lambda)} ds,$$

where $\mathbb{Q}_k^{\text{PHT}(s\Lambda)}$ is the distribution of the typical k -face of the Poisson hyperplane tessellation with intensity measure $s\Lambda$.

The preceding theorem can be rephrased by saying that the distribution of the typical k -dimensional maximal polytope of a STIT tessellation is a mixture of suitable re-scalings of the distribution of the typical k -dimensional face of Poisson hyperplane tessellations, and that the mixing distribution is a beta-distribution on $(0, t)$ with parameters d and 1.

Proof of Theorem 3. To start, let $\varphi_k : \text{MaxPolytopes}_k \rightarrow \mathbb{R}$ be a translation-invariant, non-negative measurable function for $1 \leq k \leq d-1$ and denote by $\bar{\varphi}_k(Y(t\Lambda))$ the (possibly infinite) φ_k -density of $Y(t\Lambda)$ in the sense of [23, Chap. 4.1], i.e.

$$\bar{\varphi}_k(Y(t\Lambda)) = \lim_{r \rightarrow \infty} \frac{1}{r^d \text{Vol}_d(W)} \mathbb{E} \sum_{f \in \text{MaxPolytopes}_k(Y(t\Lambda, rW))} \varphi_k(f)$$

with $W \subset \mathbb{R}^d$ some bounded convex set with positive and finite volume. The existence of this limit is guaranteed by Thm 4.1.3 ibidem. Using now Campbell's formula [23, Thm. 3.1.2] and Theorem 2 from above, we obtain, possibly with both sides infinite,

$$\begin{aligned} \bar{\varphi}_k(Y(t\Lambda)) &= \lim_{r \rightarrow \infty} \frac{1}{r^d \text{Vol}_d(W)} \mathbb{E} \sum_{f \in \text{MaxPolytopes}_k(Y(t\Lambda, rW))} \varphi_k(f) \\ &= \lim_{r \rightarrow \infty} \frac{1}{r^d \text{Vol}_d(W)} \mathbb{E} \int \varphi_k(f) \mathcal{F}_k^{Y(t\Lambda, rW)}(df) \\ &= \lim_{r \rightarrow \infty} \frac{1}{r^d \text{Vol}_d(W)} \int \varphi_k(f) \mathbb{F}_k^{Y(t\Lambda, rW)}(df) \\ &= \lim_{r \rightarrow \infty} \frac{1}{r^d \text{Vol}_d(W)} (d-k) 2^{d-k-1} \int_0^t \frac{1}{s} \left[\int \varphi_k(f) \mathbb{F}_k^{\text{PHT}(s\Lambda, rW)}(df) \right] ds \\ &= (d-k) 2^{d-k-1} \int_0^t \frac{1}{s} \left[\lim_{r \rightarrow \infty} \frac{1}{r^d \text{Vol}_d(W)} \mathbb{E} \sum_{f \in \text{Faces}_k(\text{PHT}(s\Lambda, rW))} \varphi_k(f) \right] ds \\ &= (d-k) 2^{d-k-1} \int_0^t \frac{1}{s} \bar{\varphi}_k(\text{PHT}(s\Lambda)) ds, \end{aligned} \tag{15}$$

where $\text{PHT}(s\Lambda)$ (or $\text{PHT}(s\Lambda, rW)$) is the stationary Poisson hyperplane tessellation with intensity measure $s\Lambda$ (restricted to the window rW). We use the formula (15) with $\varphi_k \equiv 1$ and denote in this case $\bar{\varphi}_k(Y(t\Lambda)) =: \lambda_k$, referred to as the *intensity of k -dimensional*

maximal polytopes. Also denote by $\lambda_k^{\text{PHT}(s\Lambda)}$ the intensity of k -faces of the Poisson hyperplane tessellation $\text{PHT}(s\Lambda)$. By stationarity, there exists a constant $C_k \in (0, \infty)$, which is independent of s , such that $\lambda_k^{\text{PHT}(s\Lambda)} = C_k s^d$. Thus, using (15), we obtain

$$\lambda_k = (d-k)2^{d-k-1} \int_0^t \frac{1}{s} C_k s^d ds = \frac{d-k}{d} 2^{d-k-1} C_k t^d = \frac{d-k}{d} 2^{d-k-1} \lambda_k^{\text{PHT}(t\Lambda)}. \quad (16)$$

For a k -dimensional polytope f , let $m(f)$ be some translation-covariant selector (for example its Steiner point or the centre of gravity). Further, letting $\mathbb{Q}_k^{\text{PHT}(s\Lambda)}$ be the distribution of the typical k -face of $\text{PHT}(s\Lambda)$ and \mathbb{Q}_k be the distribution of the typical k -dimensional maximal polytope of the STIT tessellation $Y(t\Lambda)$, we obtain, again using (15), this time with $\varphi_k(f) := \mathbf{1}[f - m(f) \in \cdot]$, together with the definition of \mathbb{Q}_k and $\mathbb{Q}_k^{\text{PHT}(s\Lambda)}$, see [23, Eq. (4.8, 4.9)],

$$\lambda_k \mathbb{Q}_k = (d-k)2^{d-k-1} \int_0^t \frac{1}{s} \lambda_k^{\text{PHT}(s\Lambda)} \mathbb{Q}_k^{\text{PHT}(s\Lambda)} ds. \quad (17)$$

Combining (17) with (16) we get, with C_k as above,

$$\begin{aligned} \mathbb{Q}_k &= (d-k)2^{d-k-1} \int_0^t \frac{1}{s} \frac{\lambda_k^{\text{PHT}(s\Lambda)}}{\lambda_k} \mathbb{Q}_k^{\text{PHT}(s\Lambda)} ds = \int_0^t \frac{d}{s} \frac{\lambda_k^{\text{PHT}(s\Lambda)}}{\lambda_k^{\text{PHT}(t\Lambda)}} \mathbb{Q}_k^{\text{PHT}(s\Lambda)} ds \\ &= \int_0^t \frac{d}{s} \frac{C_k s^d}{C_k t^d} \mathbb{Q}_k^{\text{PHT}(s\Lambda)} ds = \int_0^t \frac{ds^{d-1}}{t^d} \mathbb{Q}_k^{\text{PHT}(s\Lambda)} ds, \end{aligned}$$

which is the desired expression for \mathbb{Q}_k . \square

It is interesting to note that formally marking the k -dimensional maximal polytopes with their birth-times and repeating the argument leading to Theorem 3 with Theorem 2 replaced by its time-marked extension in Corollary 1 we obtain the birth time-marked extension of Theorem 3:

Corollary 2 *The distribution $\hat{\mathbb{Q}}_k$ of the typical birth-time-marked k -dimensional maximal polytope of $Y(t\Lambda)$ is given by*

$$\hat{\mathbb{Q}}_k = \int_0^t \frac{ds^{d-1}}{t^d} \left[\mathbb{Q}_k^{\text{PHT}(s\Lambda)} \otimes \delta_s \right] ds.$$

From these comparison theorems, mean values and even some distributional results for the typical k -dimensional maximal polytope of a stationary but anisotropic STIT tessellation can be concluded. To illustrate the general method, we exemplarily calculate at first the mean intrinsic volumes of the typical k -dimensional maximal polytope of the STIT tessellation $Y(t\Lambda)$. To neatly formulate them, let Π be the associated zonoid of the Poisson hyperplane tessellation $\text{PHT}(t\Lambda)$ in the sense of [23, Chap. 4.6].

Corollary 3 *The j -th intrinsic volume V_j (in the sense of integral geometry) of the typical k -dimensional maximal polytope I_k (with $0 \leq j \leq k \leq d-1$) is given by*

$$\mathbb{E}V_j(I_k) = \frac{\binom{d-j}{d-k}}{\binom{d}{k}} \frac{d}{d-j} \frac{V_{d-k}(\Pi)}{\text{Vol}_d(\Pi)}.$$

Proof of Corollary 3. Using Theorem 3, [23, Thm. 10.3.3] and the homogeneity of the intrinsic volumes, we get

$$\begin{aligned} \mathbb{E}V_j(I_k) &= \int_0^t \frac{ds^{d-1}}{t^d} \frac{\binom{d-j}{d-k} V_{d-j}\left(\frac{s}{t}\Pi\right)}{\binom{d}{k} \text{Vol}_d\left(\frac{s}{t}\Pi\right)} ds = \frac{\binom{d-j}{d-k}}{\binom{d}{k}} \frac{V_{d-k}(\Pi)}{\text{Vol}_d(\Pi)} \int_0^t \frac{ds^{d-1}}{t^d} \frac{\left(\frac{s}{t}\right)^{d-j}}{\left(\frac{s}{t}\right)^d} ds \\ &= \frac{\binom{d-j}{d-k}}{\binom{d}{k}} \frac{d}{d-j} \frac{V_{d-k}(\Pi)}{\text{Vol}_d(\Pi)}, \end{aligned}$$

which completes the proof. \square

Note that in the isotropic case, i.e. when \mathcal{R} is the uniform distribution on \mathcal{S}_{d-1} in the factorization (2), the zonoid Π is a d -dimensional ball with radius proportional to t . More specifically, we have in this case

$$\mathbb{E}V_j(I_k) = \frac{d}{(d-j)\kappa_j} \binom{k}{j} \left(\frac{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \right)^j \frac{1}{t^j}.$$

As a second example we turn now to the length distribution of the typical I-segment, which is nothing than the maximal polytope of dimension one.

Corollary 4 *The distribution of the length of the typical I-segment of a stationary and isotropic STIT tessellation with time parameter $t > 0$ is a mixture of exponential distributions with parameter $\gamma_1 s$. The mixing distribution is a beta-distribution on $(0, t)$ with parameters d and 1. Its density with respect to the Lebesgue measure on $(0, \infty)$ is given by*

$$p_d(x) = \int_0^t \gamma_1 s e^{-\gamma_1 s x} \frac{ds^{d-1}}{t^d} ds = \frac{d}{(\gamma_1 t)^d x^{d+1}} \Gamma(d+1, \gamma_1 t x),$$

where $\Gamma(\cdot, \cdot)$ is the lower incomplete Gamma-function and $\gamma_1 = \Gamma(d/2)/(\Gamma(1/2)\Gamma((d+1)/2))$.

Proof of Corollary 4. This follows immediately from Theorem 3 and the well known fact that the length distribution of the typical edge of a stationary and isotropic Poisson hyperplane tessellation with intensity $0 < s < t$ is an exponential distribution with parameter $\gamma_1 s$. \square

In particular for $d = 2$ and $d = 3$ we have the densities

$$\begin{aligned} p_2(x) &= \frac{1}{t^2 x^3} \left(\pi^2 - (\pi^2 + 2\pi t x + 2t^2 x^2) e^{-\frac{2}{\pi} t x} \right), \\ p_3(x) &= \frac{3}{t^3 x^4} \left(48 - (48 + 24t x + 6t^2 x^2 + t^3 x^3) e^{-\frac{1}{2} t x} \right). \end{aligned}$$

The mean segment lengths are π/t in the planar case (see also [21]) and $3/t$ for $d = 3$ (compare with [32]). Moreover, the variance of the length of the typical I-segment in the spatial case is given by $24/t^2$, which was not known before. In general, from the explicit length density formula it is easily seen that for the length of the typical I-segment only the moments of order 1 up to $d - 1$ exist.

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